ISRAEL JOURNAL OF MATHEMATICS, Vol. 67, No. 3, 1989

ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR PIECEWISE EXPANDING C^2 TRANSFORMATIONS IN R^N

BY

P. GÓRA[a] AND A. BOYARSKY[b,†]

[a]*Department of Mathematics, Warsaw University, Warsaw, Poland; and*
[b]*Department of Mathematics, Concordia University, 7141 Sherbrooke St. West, Montreal, Canada H4B 1R6*

ABSTRACT

Let S be a bounded region in R^N and let $\mathscr{P} = \{S_i\}_{i=1}^m$ be a partition of S into a finite number of subsets having piecewise C^2 boundaries. We assume that where C^2 segments of the boundaries meet, the angle subtended by tangents to these segments at the point of contact is bounded away from 0. Let $\tau : S \to S$ be piecewise C^2 on $\mathscr{P}$ and expanding in the sense that there exists $0 < \sigma < 1$ such that for any $i = 1, 2, \ldots, m$, $\| D\tau_i^{-1} \| < \sigma$, where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\| \quad \|$ is the euclidean matrix norm. The main result provides an upper bound on σ which guarantees the existence of an absolutely continuous invariant measure for τ.

1. Introduction

In 1973 Lasota and Yorke [14] proved a general sufficient condition for the existence of an absolutely continuous invariant measure (a.c.i.m.) for expanding, piecewise C^2 transformations on the interval. In spite of the suggestion at the end of [14] that the "bounded variation" techniques of [14] can be easily used to obtain analogous results in higher dimensions, the generalization of the main result of [14] has taken much longer than expected. This was partly due to the difficulty in finding the right definition of variation in higher dimensions. For smooth maps on boundaryless domains, general results for the existence of a.c.i.m. were known as early as 1969 [12]. For piecewise C^2 maps in R^N, the first major attempt to prove an existence result came in 1979 [11].

† The research of the second author was supported by NSERC and FCAR grants.
Received March 7, 1989

The authors do not use a bounded variation argument but the proof, based on a one-dimensional version [10], is flawed. The first correct, but partial result, appeared in [8]. There, the author considers expanding, piecewise analytic transformations on the unit square partitioned by smooth boundaries. A complicated definition of bounded variation is used and the method cannot be extended beyond dimension 2. For boundaries which are not analytic, the sufficient condition that arises is rather complicated [9].

Working on rectangular partitions and with expanding, piecewise C^2 transformations which are very restrictive (the ith component of the transformation depends only on the ith variable), Jabłoński [7] proved the existence of an a.c.i.m. using the Tonnelli definition of bounded variation. The technique in this special setting is exactly analogous to that in [14].

In [17] a necessary and sufficient condition for the existence of a.c.i.m. is presented, but in most cases it cannot be applied.

With the publication of [3], a major new tool became available. The definition of variation of a function in R^N as the integral of its generalized derivative [3] led to the following partial result [1]: piecewise C^2 transformations on a rectangular partition satisfying a strong expansiveness condition (which depends on the dimension N of the space) have an a.c.i.m. Another partial result was obtained in [6].

In this note we follow through the approach of [1] in a more general setting. With only C^2 restrictions on the boundaries of the partition and with a mild restriction on how these boundaries meet, we prove the existence of an absolutely continuous invariant measure for τ if the slope of τ is sufficiently large.

In this setting, we can invoke the powerful Ionescu Tulcea and Marinescu Theorem [5] to obtain a useful spectral decomposition for the Perron–Frobenius operator of τ and, as a consequence, prove strong ergodic properties of the transformation itself.

Applications of ergodic theory for higher dimensional transformations can be found in [16, 18].

2. Main result

Let S be a bounded region in R^N and let τ be a transformation from S into S. We assume that τ is piecewise C^2 and expanding, i.e.,

(a) there exists a partition $\mathscr{P} = \{S_i\}_{i=1}^m$ of S, where m is a positive integer, and each S_i is a bounded closed domain having a piecewise C^2 boundary of finite $(N-1)$-dimensional measure;

(b) $\tau_i = \tau_{|S_i}$ is a C^2, 1–1 transformation from $\mathrm{int}(S_i)$ onto its image and can be extended as a C^2 transformation onto S_i, $i = 1, 2, \ldots, m$;

(c) there exists $0 < \sigma < 1$ such that for any $i = 1, 2, \ldots, m$,

$$\| D\tau_i^{-1} \| < \sigma, \tag{1}$$

where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\| \quad \|$ is the euclidean matrix norm.

We remark that condition (1) implies, for $\tau_i^{-1}(x)$, $\tau_i^{-1}(y)$ close enough,

$$\rho(\tau_i^{-1}(x), \tau_i^{-1}(y)) < \sigma\rho(x, y),$$

where $x, y \in R_i \equiv \tau(\mathrm{int}(S_i))$ and ρ is the euclidean metric in R^N.

Condition (1) is implied by any of the following equivalent conditions:

(c1) all the eigenvalues of $D\tau_i^{-1}$ are smaller than 1;

(c2) all the eigenvalues of $D\tau_i$ are larger than 1.

If $|\partial\tau_{ij}^{-n}/\partial x_k| < 1$ for some n, where τ_{ij}^{-1} is the jth component of τ_i^{-1}, for $i = 1, \ldots, m$, and $1 \leqq j, k \leqq N$, then condition (c) is true for some iterate τ^l.

Let $Z = \bigcup_{i=1}^m \mathrm{int}(S_i)$. We will consider τ as a transformation from Z into S. Our assumptions imply it is nonsingular, i.e., $\lambda_N(\tau^{-1}(\quad))$ is absolutely continuous with respect to Lebesgue measure λ_N on S. This is enough for τ to induce the Perron–Frobenius operator

$$P_\tau : L_1(S) \to L_1(S),$$

defined by

$$P_\tau f(x) = \sum_{i=1}^m \frac{f(\tau_i^{-1}(x))}{\mathcal{J}(\tau_i^{-1}(x))} \chi_{R_i}(x),$$

where χ_R is the characteristic function of the set R and $\mathcal{J}(\beta)$ is the absolute value of the Jacobian of β. The properties of P_τ are described in [13], for example. It is well known that f is a τ-invariant density if and only if $P_\tau f = f$.

The main tool of the paper is the multidimensional notion of variation defined using derivatives in the distributional sense [3]:

$$V(f) = \int_{R^N} \| Df \| = \sup\left\{ \int_{R^N} f \,\mathrm{div}(g) d\lambda_N : g = (g_1, \ldots, g_N) \in C_0^1(R^N, R^N) \right\},$$

where $f \in L_1(R^N)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(R^N, R^N)$ is the space of continuously differentiable functions from R^N into R^N having compact support. We will use the following property of variation which is easily derived from [3, Remark 2.14]:

If $f = 0$ outside a closed domain A whose boundary is Lipschitz continuous, $f_{|A}$ is continuous, $f_{|\mathrm{int}(A)}$ is C^1, then

$$V(f) = \int_{\mathrm{int}(A)} \| Df \| \, d\lambda_N + \int_{\partial A} |f| \, d\lambda_{N-1},$$

where λ_{N-1} is the $(N-1)$-dimensional measure on the boundary of A.

In the sequel we shall consider the Banach space [3, Remark 1.12],

$$BV(S) = \{ f \in L_1(S) : V(f) < +\infty \},$$

with the norm $\| f \|_{BV} = \| f \|_{L_1} + V(f)$.

Before stating the main theorem, we shall need a number of lemmas.

Consider an element $S_i \in \mathscr{P}$. Let x be a point in ∂S_i and $y = \tau(x)$ a point in $\partial(\tau(S_i))$. Let $\mathscr{J}$ be the Jacobian of $\tau_{|S_i}$ at x and $\mathscr{J}_0$ the Jacobian of $\tau_{|\partial(S_i)}$ at x.

LEMMA 1. $\mathscr{J}_0/\mathscr{J} \leqq \sigma$.

PROOF. Let C_n be a neighbourhood of y in $\tau(S_i)$ and $B_n = C_n \cap \partial(\tau(S_i))$, $n = 1, 2, \ldots$. Let γ be a curve perpendicular to $\partial(\tau(S_i))$ at y extending into C_n, and let $\gamma_n = \gamma \cap C_n$. We foliate C_n into hypersurfaces $B_n(t)$, $t \in \gamma_n$, each $B_n(t)$ being perpendicular to γ_n, and thus approximately parallel to B_n. We assume that for any n, $\lambda_{N-1}(B_n(t)) = \lambda_{N-1}(B_n)$ for all $t \in \gamma_n$. Then, if C_n is small enough, we have:

$$\lambda_N(C_n) = (1 + \varepsilon_n) \int_{\gamma_n} \lambda_{N-1}(B_n(t)) d\gamma_n(t) = (1 + \varepsilon_n) \lambda_{N-1}(B_n) \lambda_1(\gamma_n),$$

where $\varepsilon_n \to 0$ as $\mathrm{diam}(C_n) \to 0$. On the other hand, we have

$$1/\mathscr{J} = \lim_{\mathrm{diam}(C_n) \to 0} \lambda_N(\tau^{-1}(C_n))/\lambda_N(C_n).$$

To estimate $\lambda_N(\tau^{-1}(C_n))$, let $\eta_n = \tau^{-1}(\gamma_n)$, $D_n(t) = \tau^{-1}(B_n(t))$, $t \in \gamma_n$. Let $\mathscr{J}_0(t, \zeta)$ be the Jacobian of $\tau_{|D_n(t)}$, where $t \in \gamma_n$, $\zeta \in D_n(t)$. Since $\tau_{|S_i}$ is a C^2-diffeomorphism, $1/\mathscr{J}_0(t, \zeta)$ is a C^1-function. Thus

$$\begin{aligned} \lambda_{N-1}(D_n(t)) &= \int_{B_n(t)} (1/\mathscr{J}_0(t, \zeta)) d\lambda_{N-1}(\zeta) \\ &\leqq \int_{B_n(t)} (1/\mathscr{J}_0 + K \, \mathrm{diam}(C_n)) d\lambda_{N-1}(\zeta) \\ &= (1/\mathscr{J}_0 + K \, \mathrm{diam}(C_n)) \lambda_{N-1}(B_n), \end{aligned}$$

for a constant $K > 0$.

We have

$$\lambda_N(\tau^{-1}(C_n)) \leqq \int_{\eta_n} \lambda_{N-1}(D_n(t))d\eta_n(t)$$
$$= (1/\mathcal{J}_0 + K\,\mathrm{diam}(C_n))\lambda_{N-1}(B_n)\lambda_1(\eta_n)$$
$$= \sigma(1/\mathcal{J}_0 + K\,\mathrm{diam}(C_n))\lambda_{N-1}(B_n)\lambda_1(\gamma_n).$$

Thus

$$\lambda_N(\tau^{-1}(C_n))/\lambda_N(C_n) \leqq \sigma(1/\mathcal{J}_0 + K\,\mathrm{diam}(C_n))(1+\varepsilon_n).$$

Taking the limit as $\mathrm{diam}(C_n) \to 0$, we get $1/\mathcal{J} \leqq \sigma/\mathcal{J}_0$. ■

We note that this result is a considerable improvement over the condition $\mathcal{J}_0/\mathcal{J} \leqq N\sigma$ derived in [1] for transformations on rectangular partitions.

Let S be a closed domain in R^N with $W = \partial S$, which is piecewise C^2 and of finite $(N-1)$-dimensional measure. Let D denote the set of singular points of W, and let $\nu(x)$ denote the normalized outward normal vector at x (if $x \in D$, there are several possible outward normals). For any $x \in W$, let W_x be a small neighbourhood of x in W, contained completely in one face of W. If $x \in D$, we use a half-neighbourhood which is in one face.

For any $x \in W$, we define an R^N-neighbourhood of x, $\mathcal{U}(\delta, \alpha, x)$, $\delta > 0$, $\pi/2 < \alpha < \pi$, as follows: let $H(y)$, $y \in W_x$ be a C^1 normalized vector field, such that $\angle(H(x), \nu(x)) = \alpha$ ($\angle$ denotes angle). For any point $y \in W_x$, let $L_y = [y, y + \delta H(y)]$, the segment joining y and $y + \delta H(y)$. Now, let $\mathcal{U}(\delta, \alpha, x) = \bigcup_{y \in W_x} L_y$. If W_x and δ are small enough $\mathcal{U}(\delta, \alpha, x)$ lies completely on one side of W_x.

Lemma 2. *For any $x \in W$, and for any $\varepsilon > 0$ sufficiently small, we can choose W_x so that:*

$$\frac{(1+\varepsilon)^2}{|\cos\alpha| - \varepsilon} \int_{\mathcal{U}(\delta,\alpha,x)} f\,d\lambda_N \geqq \int_{W_x} \left(\int_{L_y} f(\xi)d\xi \right) d\lambda_{N-1}(y) \tag{2}$$

for any $f \in C^1(R^N)$.

Proof. The inequality (2) obviously holds if W_x is a piece of an $(N-1)$-dimensional hyperplane. The idea of the proof is to convert our situation to that simple case.

By an orthogonal change of variables to the variables $(z_1, \ldots, z_N)$ in R^N, we can ensure that the hyperplane T_x tangent to W_x at x is given by $z_N = 0$, and the angle α between $\nu(x)$ and L_x is contained in the plane $z_1 = z_2 = \cdots = z_{N-2} = 0$. We choose W_x so small that it can be described by the equation: $z_N =$

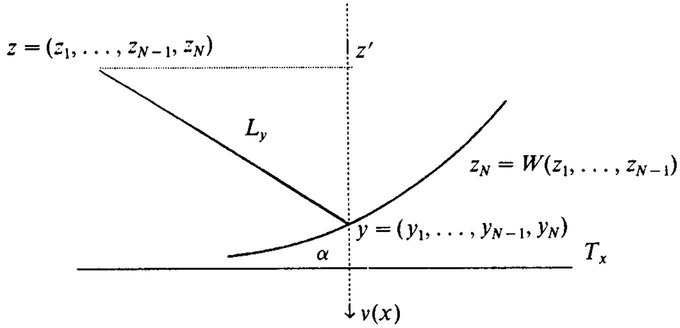

Fig. 1.

$W(z_1, \ldots, z_N)$; see Fig. 1. First, we straighten out all the segments L_y. We shall do this by keeping the same point y and shifting the point z to z', as shown in Fig. 1. If $z = (z_1, \ldots, z_N)$ is any point on the segment L_y, $y = (y_1, \ldots, y_N)$, then:

$$z = (y_1, \ldots, y_N) + |\operatorname{tg} \alpha|(z_N - y_N)[h_1, \ldots, h_N]$$

and $z' = (y_1, \ldots, y_{N-1}, z_N)$, where $[h_1, \ldots, h_N] = H(y)$, $\alpha = \alpha(y_1, \ldots, y_N) = \angle(H(y), v(x))$. Thus the straightening out is accomplished by the transformation φ_1 defined by:

$$\varphi_1(z_1, \ldots, z_N) = (z_1, \ldots, z_N) - |\operatorname{tg} \alpha|(z_N - y_N)[h_1, \ldots, h_{N-1}, 0],$$

where α, $h_1, \ldots, h_{N-1}$ are C^1 functions of $y_1, \ldots, y_N$, $y_N = W(y_1, \ldots, y_{N-1})$ and $y_1, \ldots, y_{N-1}$ are C^2 functions of $z_1, \ldots, z_N$.

Let

$$\mathcal{J}(\varphi_1) = \left| \det \left(\frac{\partial \varphi_{1,i}}{\partial z_j} \right)_{i,j=1}^{N} \right|$$

be the Jacobian of φ_1.

Notice that for $z = x = y$ we have $h_1 = h_2 = \cdots = h_{N-2} = \partial W / \partial z_1 = \partial W / \partial z_2 = \partial W / \partial z_{N-1} = z_N - y_N = 0$ and that the derivatives of all involved functions are continuous and bounded. This implies that $\mathcal{J}(\varphi_1)(x) = 1$ and that choosing W_x and δ small enough we can ensure $\mathcal{J}(\varphi_1) \leqq 1 + \varepsilon$ on $\mathcal{U}(\delta, \alpha, x)$.

We now straighten out the surface W_x. This is done by the transformation

$$\varphi_2(z_1, \ldots, z_N) = (z_1, \ldots, z_{N-1}, z_N - W(z_1, \ldots, z_{N-1})).$$

The Jacobian of φ_2, $\mathcal{J}(\varphi_2)$, is equal to 1. Let $\mathcal{U} = \varphi_2 \circ \varphi_1(\mathcal{U}(\delta, \alpha, x))$ and $W'_x = \varphi_2\varphi_1(W_x)$. Then, we have:

$$\int_{\mathcal{U}(\delta,\alpha,x)} f\,d\lambda_N = \int_{\mathcal{U}} f \frac{1}{J(\varphi_2)} \frac{1}{J(\varphi_1)}\,d\lambda_N \geqq \frac{1}{(1+\varepsilon)} \int_{\mathcal{U}} f\,d\lambda_N$$

$$= \frac{1}{(1+\varepsilon)} \int_{W'_x} \left(\int_{\varphi_2(\varphi_1(L_y))} f(\eta)d\eta \right) d\lambda_{N-1}(w),$$

where $\eta = \varphi_2 \circ \varphi_1(\xi)$ and $w = \varphi_2(y)$. To obtain (2), we need $d\eta/d\xi$ and $J(\bar{\varphi}_2)$, where $\bar{\varphi}_2$ *is* φ_2 treated as a transformation from W_x into W'_x. From Fig. 1, it is easy to see that $d\eta/d\xi = |\cos\alpha|$, where $\alpha = \alpha(z_1, \ldots, z_N)$. If W_x is small enough $|\cos\alpha(z_1, \ldots, z_N)| \geqq |\cos\alpha| - \varepsilon$. Also,

$$J(\bar{\varphi}_2)(y) = |\operatorname{Det}(\langle g_i, g_j \rangle)_{i,j=1}^{N-1}|^{-1},$$

where $g(z_1, \ldots, z_{N-1}) = (z_1, \ldots, z_{N-1}, W(z_1, \ldots, z_{N-1}))$ and $g_i = \partial g/\partial z_i$, $i = 1, \ldots, N-1$. Since this determinant is 1 at the point x and all derivatives involved are at least continuous, we can choose W_x so small that

$$J(\bar{\varphi}_2)(y) \geqq \frac{1}{1+\varepsilon}, \qquad \text{for } y \in W_x.$$

We obtain

$$\int_{\mathcal{U}(\delta,\alpha,x)} f\,d\lambda_N \geqq \frac{1}{(1+\varepsilon)} \int_{W_x} \left(\int_{L_y} f(\xi)|\cos\alpha|\,d\xi \right) J(\bar{\varphi}_2)(y)d\lambda_{N-1}(y)$$

$$\geqq \frac{|\cos\alpha| - \varepsilon}{(1+\varepsilon)^2} \int_{W_x} \left(\int_{L_y} f(\xi)d\xi \right) d\lambda_{N-1}(y). \qquad \blacksquare$$

By a regular cone in R^N we mean a cone whose base is a $(N-1)$-dimensional disk B and such that the central ray L joining the vertex to the center of the disk B is perpendicular to the disk. We define the angle subtended at the vertex of a regular cone to be the angle between L and any line joining the vertex to a point on the boundary of B.

Let S, like before, be a closed domain in R^N having piecewise C^2 boundary of finite $(N-1)$-dimensional measure.

Let us now construct at any singular point $x \in D$, the largest possible regular cone having its vertex at x and which lies completely in S. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S) = \min_{x \in D} \theta(x).$$

Since the faces of W meet at angles bounded away from 0, $\beta(S) > 0$. Let $\alpha(S) = \pi/2 + \beta(S)$ and

$$a(S) = |\cos \alpha(S)|.$$

Now we will construct a C^1 field of segments L_y, $y \in W = \partial S$, every L_y being a central ray of a regular cone contained in S, with angle subtended at the vertex y greater than or equal to $\beta(S)$.

We start at the points $y \in D$, where the minimal angle $\beta(S)$ is attained, defining L_y to be central rays of the largest regular cones contained in S. Then we extend this field of segments to the C^1 field we want, making L_y short enough to avoid overlapping. Let $\delta(y)$ be the length of L_y, $y \in W$. By compactness of W we have

$$\delta(S) = \inf_{y \in W} \delta(y) > 0.$$

Now we shorten L_y of our field, making them all of the length $\delta(S)$.

Lemma 3. *If some S is a closed domain with piecewise C^2 boundary of finite $(N-1)$-dimensional measure, whose smooth faces meet at angles bounded away from zero, and f is a C^1 function on S, then*

$$\int_{\partial S} f(y) d\lambda_{N-1}(y) \leqq \frac{1}{a(S)} \left(\frac{1}{\delta(S)} \int_S f \, d\lambda_N + V(f, \operatorname{int}(S)) \right).$$

Proof. Fix $\varepsilon > 0$, sufficiently small. We partition $W = \partial S$ into sets W_{x_i}, $i = 1, \ldots, M$, for which inequality (2) holds and define sets $\mathscr{U}(\sigma(S), \alpha(S), x_i)$, $i = 1, \ldots, M$, using the field of segments L_y, $y \in W$, constructed above. Now, for any $y \in W \setminus D$, we have:

$$f(y) \leqq \min\{f(x) : x \in L_y\} + V_{L_y}(f),$$

and

$$f(y) \leqq \frac{1}{|L_y|} \int_{L_y} f(\xi) d\xi + V_{L_y}(f) \leqq \frac{1}{\delta(S)} \int_{L_y} f(\xi) d\xi + V_{L_y}(f),$$

where $V_L(f)$ is the variation of f along the line L_y. We also have

$$V_{L_y}(f) \leqq \int_{L_y} \| Df(x) \| \, dx.$$

Integrating over W_{x_i}, $i = 1, \dots, M$, we get

$$\int_{W_x} f(y)d\lambda_{N-1}(y)$$

$$\leqq \frac{(1+\varepsilon)^2}{a(S)-\varepsilon}\frac{1}{\delta(S)}\int_{\mathscr{U}(\delta(S),\alpha(S),x_i)} f\,d\lambda_N + \frac{(1+\varepsilon)^2}{a(S)-\varepsilon}\int_{\mathscr{U}(\delta(S),\alpha(S),x_i)} \| Df \|\, d\lambda_N.$$

Summing up, and noting that $\mathscr{U}(\delta(S), \alpha(S), x_i)$ do not overlap, we get:

$$\int_{\partial S} f(y)d\lambda_{N-1}(y) \leqq \frac{(1+\varepsilon)^2}{a(S)-\varepsilon}\left(\frac{1}{\delta(S)}\int_S f\,d\lambda_N + \int_S \| Df \|\, d\lambda_N\right).$$

Since ε is arbitrary, Lemma 3 is proved. ∎

Let $\tau : S \to S$ be a piecewise C^2 expanding transformation. We assume that the sets S_i, $i = 1, \dots, m$, of its defining partition have piecewise C^2 boundaries of finite $(N-1)$-dimensional measure and that

$$a = \min\{a(S_i) : i = 1, \dots, m\} > 0.$$

Let

$$\delta = \min\{\delta(S_i) : i = 1, \dots, m\} > 0.$$

Under these assumptions, we prove the following results.

Lemma 4. *Let $f \in L_1(S)$. If $V(f) < +\infty$, then*

$$V(P_\tau f) \leqq \sigma(1 + 1/a)V(f) + K \| f \|_{L_1},$$

for some constant $K < +\infty$.

Proof. First we assume that $f \in C^1(S)$. Then

$$P_\tau f = \sum_{i=1}^{m} \frac{f(\tau_i^{-1})}{\mathscr{J}(\tau_i^{-1})}\chi_{R_i}.$$

Let $F_i = f(\tau_i^{-1})/\mathscr{J}(\tau_i^{-1})$, $i = 1, \dots, m$. Then,

$$\int_{R^N} \| DP_\tau f \|\, d\lambda_N \leqq \sum_{i=1}^{m} \int_{R^N} \| D(F_i\chi_{R_i}) \|\, d\lambda_N$$

$$\leqq \sum_{i=1}^{m}\left(\int_{R^N} \| (DF_i)\chi_{R_i} \|\, d\lambda_N + \int_{R^N} \| F_i(D\chi_{R_i}) \|\, d\lambda_N\right).$$

We have

$$\int_{R^N} \| (DF_i)\chi_{R_i} \| \, d\lambda_N$$

$$= \int_{R_i} \| (DF_i) \| \, d\lambda_N$$

$$\leqq \int_{R_i} \| (Df(\tau_i^{-1}))/\mathscr{J}(\tau_i^{-1}) \| \, d\lambda_N + \int_{R_i} \| f(\tau_i^{-1}) D(1/\mathscr{J}(\tau_i^{-1})) \| \, d\lambda_N$$

$$\leqq \sum_{j=1}^{N} \int_{R_i} \frac{\left| \frac{\partial f}{\partial x_j}(\tau_i^{-1}) \right| \left| \sum_{k=1}^{N} \frac{\partial \tau_{ij}^{-1}}{\partial x_k} \right|}{\mathscr{J}(\tau_i^{-1})} \, d\lambda_N + \int_{R_i} | f(\tau_i^{-1}) | \frac{K_0}{\mathscr{J}(\tau_i^{-1})} \, d\lambda_N$$

$$\leqq \sigma \int_{S_i} \| Df \| \, d\lambda_N + K_0 \int_{S_i} \| f \| \, d\lambda_N,$$

where τ_{ij}^{-1} is the jth component of τ_i^{-1}, $1 \leqq i \leqq m$, $1 \leqq j \leqq N$, and K_0 is an upper bound for $\| D(\mathscr{J}(\tau_i^{-1}))^{-1} \| \mathscr{J}(\tau_i^{-1})$, which exists by the C^2 assumption on τ_i, $1 \leqq i \leqq m$.

Now, using Example 1.4 of [3], we obtain:

$$\int_{R^N} \| F_i(D\chi_{R_i}) \| \, d\lambda_N = \int_{\partial R_i} | F_i | \, d\lambda_{N-1}$$

$$= \int_{\partial R_i} | f(\tau_i^{-1}) | \mathscr{J}(\tau_i^{-1})^{-1} d\lambda_{N-1} = \int_{\partial S_i} | f | (\mathscr{J}_0/\mathscr{J}) \, d\lambda_{N-1},$$

where $\mathscr{J}_0$ is the Jacobian of $\tau_i : \partial S_i \to \partial R_i$.

By Lemma 1, $\mathscr{J}_0/\mathscr{J} \leqq \sigma$. Therefore, using Lemma 3, we obtain:

$$\int_{R^N} \| F_i(D\chi_{R_i}) \| \, d\lambda_N \leqq \sigma \int_{\partial S_i} | f | \, d\lambda_{N-1}$$

$$\leqq \frac{\sigma}{a} V(f, S_i) + \frac{\sigma}{a\delta} \int_{S_i} | f | \, d\lambda_N.$$

Summing up, we get:

$$V(P_\tau f) \leqq \sigma(1 + 1/a) V(f) + (K_0 + \sigma/a\delta) \| f \|_{L_1}.$$

Let $K = K_0 + \sigma/a\delta$.

In general, let $f \in BV(S)$. There exists a sequence of C^1 functions f_n, $n = 1, 2, \ldots$, which approximates f in $BV(S)$. We have:

$$
\begin{aligned}
V(P_\tau f) &= \sup_h \int_{R^N} (P_\tau f)\operatorname{div}(h)d\lambda_N \\
&= \sup_h \lim_{n\to+\infty} \int_{R^N} (P_\tau f_n)\operatorname{div}(h)d\lambda_N \\
&= \sup_h \limsup_{n\to+\infty} \int_{R^N} (P_\tau f_n)\operatorname{div}(h)d\lambda_N \\
&\leqq \limsup_{n\to+\infty} V(P_\tau f_n) \\
&\leqq \lim_{n\to+\infty} (\sigma(1+1/a)V(f_n) + K \| f_n \|_{L_1}) \\
&= \sigma(1+1/a)V(f) + K \| f \|_{L_1},
\end{aligned}
$$

where $h \in C^1(R^N, R^N)$, $\| h \| \leqq 1$. ■

Lemma 5. *For any $f \in BV(S)$*

$$
\| P_\tau f \|_{BV} \leqq \sigma(1+1/a) \| f \|_{BV} + (K+1) \| f \|_{L_1}. \tag{3}
$$

Proof. Follows directly from Lemma 4 and the definition of $\| \ \|_{BV}$. ■

Remark. For $N = 1$, the inequality (3) yields the same slope condition as in the original Lasota–Yorke Theorem [14].

We can now state the main result of this paper.

Theorem 1. *Let $\tau : S \to S$, $S \subset R^N$, be a piecewise C^2, expanding transformation. If $\sigma(1+1/a) < 1$, then τ admits an absolutely continuous invariant measure.*

Proof. From inequality (3) it follows that the set $\{ \| P_\tau^i(1) \|_{BV} \}_{i \geqq 1}$ is uniformly bounded. Hence the set $\{P_\tau^i(1)\}_{i \geqq 1}$ is weakly compact in L_1 (actually it is strongly compact), and it follows from the Kakutani–Yoshida Theorem that P_τ has a nontrivial fixed point f^* which is the density of an a.c.i.m. ■

Corollary 1. *Let $\tau : S \to S$, $S \subset R^N$, be piecewise C^2 and such that some iterate τ^k satisfies $\sigma(1+1/a) < 1$ (σ and a corresponds to τ^k). then τ admits an a.c.i.m.*

Proof. Straightforward. ■

Example. For a rectangular partition of a rectangular domain in R^N, we have $1/a = \sqrt{N}$, which gives the expansion condition $\sigma(1+\sqrt{N}) < 1$.

REMARK. The expansion condition $\sigma < (1 + 1/a)^{-1}$ depends only on the domain S and not on the transformation τ. Under certain conditions on τ, we can obtain an improved expansion condition such as is done in [9] in dimension 2. These conditions are usually very complex and require accurate knowledge of the transformation.

3. Spectral decomposition

In this section we will use the Ionescu Tulcea and Marinescu Theorem [5]. First we have to check that the assumptions of the theorem are satisfied:

We consider the space BV, $\| \ \|_{BV}$ as included in L_1, $\| \ \|_{L_1}$.

(1) By the semicontinuity property of variation (Theorem 1.9 of [3]), if $\{f_n\} \in BV$, $\| f_n \|_{BV} \leqq D$, for $n = 1, 2, \ldots$ and $f_n \to f$ in L_1, then $f \in BV$ and $\| f \|_{BV} \leqq D$.

(2) The operator norm of the Perron–Frobenius operator P_τ is 1.

(3) There exist constants $R > 0$, $0 < r < 1$ such that

$$\| P_\tau f \|_{BV} \leqq r \| f \|_{BV} + R \| f \|_{L_1}, \qquad \text{for } f \in BV.$$

This follows by Lemma 5 if $\sigma(1 + 1/a) < 1$.

(4) The image of any bounded subset of BV under the Perron–Frobenius operator is relatively compact in L_1. This follows from the compactness Theorem 1.19 of [3].

The Ionescu Tulcea and Marinescu Theorem implies the following result:

THEOREM 2. *Let $\tau : s \to S$, $S \subset R^N$, $N \geqq 1$, be a piecewise C^2 and expanding transformation with $\sigma(1 + 1/a) < 1$ and let $P = P_\tau$ be its Perron–Frobenius operator. Then:*

(a) *P (as an operator from BV into BV) has a finite number of eigenvalues of modulus 1: $\alpha_1, \ldots, \alpha_t$. They are roots of unity and*

$$P = \sum_{i=1}^{t} \alpha_i P_i + T,$$

where $P_i : BV \to BV$ are linear projections with finite dimensional range, and $T : BV \to BV$ is a continuous linear operator;

(b) *$P_i^2 = P_i$, $P_i P_j = 0$ $(i \neq j)$, $P_i T = T P_i = 0$, $1 \leqq i, j \leqq t$;*

(c) *$\| T^n \|_{BV} \leqq M/(1 + h)^n$, $n = 1, 2, \ldots$, for some M, $h > 0$.*

REMARK (see [15]). Operators P_i, $i = 1, \ldots, t$ and T have unique extensions onto L_1. Moreover $P_i(L_1) \subset BV$, $\| P_i \|_{L_1} \leqq 1$ and $\sup_n \| T^n \| < +\infty$. For any $f \in L_1$, $T^n f \to 0$ in L_1, as $n \to +\infty$.

The following theorem and corollaries are consequences of the representation of the Perron–Frobenius operator obtained in Theorem 2.

THEOREM 3 (see [15]). *Assume that* 1 *is the only eigenvalue of* P *with modulus* 1 (*we can consider* P^k, *where* k *is the smallest common multiplier of orders of* $\alpha_1, \dots, \alpha_t$). *Let* $Ug = g \circ \tau$, *for* $g \in L_\infty$. *Then there exist nonnegative functions* $\phi_1, \dots, \phi_s \in BV$ *and* $\psi_1, \dots, \psi_s \in L_\infty$ *such that*:

(a) *For any* $f \in L_1$

$$P_1 f = \sum_{i=1}^{s} \left(\int_{R^N} f \psi_i d\lambda_N \right) \phi_i,$$

(b) $P\phi_i = \phi_i$, $U\psi_i = \psi_i$, $i = 1, \dots, s$.

(c) $\int_{R^N} \phi_i \psi_i d\lambda_N = \delta_{ij}$, $\inf\{\phi_i, \phi_j\} = 0 = \inf\{\psi_i, \psi_j\}$ *as* $i \neq j$, *and* $\int_{R^N} \phi_i d\lambda_N = 1$, $1 \leqq i, j \leqq s$,

(d) *There exist measurable sets* $C_1, \dots, C_s \subset S$ *such that* $\psi_i = \chi_{C_i}$ *a.e., for* $i = 1, \dots, s$ *and* $S = \bigcup_{i=1}^{s} C_i$ *a.e.*

(e) $\bigcap_{n=1}^{\infty} U^n(L_1) = \bigcap_{n=1}^{\infty} U^n(L_\infty) = \mathrm{Span}\{\psi_1, \dots, \psi_s\}$.

(f) *For any* $f \in L_1$, $U^n f \to P_1^* f$ *in* $\sigma(L_1, BV)$-*topology*; *for any* $f \in L_\infty$, $U^n f \to P_1^* f$ *in* $\sigma(L_\infty, L_1)$-*topology*;

$$P_1^* f = \sum_{i=1}^{s} \left(\int_{R^N} f \phi_i d\lambda_N \right) \psi_i.$$

COROLLARY 2 (see [15]). *For any* $1 \leqq i \leqq s$, $\tau^k_{|C_i}$ *is an exact transformation*.

COROLLARY 3 (see [15]). *If we assume that* τ *is mixing* (*or weakly mixing, which is equivalent in this situation*), *and* μ *is its unique a.c.i.m., then* τ *has the property of exponential decay of correlation*: *Let* $f \in BV$, $g \in L_\infty$ *and* $\mu(f) = \int_{R^N} f\, d\mu$, $\mu(g) = \int_{R^N} g\, d\mu$. *Then*

$$\int_{R^N} (fg(\tau^n) - \mu(f)\mu(g)) d\mu \leqq \mu(f) V(f) \| g \|_{L_\infty} r^n, \qquad n = 1, 2, \dots,$$

where $0 < r < 1$ *is the constant of condition* (3).

COROLLARY 4 (see [15]). *If we assume that* τ *is mixing, then the defining partition* $\{S_i\}_{i=1}^{m}$ *is weakly Bernoulli for* τ, *whch implies that the natural extension of the dynamical system* (τ, μ) *is isomorphic to a Bernoulli shift* (μ *is the* τ-*invariant absolutely continuous measure*).

COROLLARY 5 (see [4]). *Assume that τ, μ is weakly mixing (it is equivalent to being mixing or exact in our situation). Let $f \in BV$ and $\mu(f) = \int_{R^N} f\, d\mu = 0$. Define*

$$S(t) = \sum_{i=0}^{t-1} f \circ \tau^i,$$

which is a stochastic process on (S, μ). Then the series (σ below has nothing to do with the contraction constant of formula (1), we use it here only for historical reasons)

$$\sigma^2 = \int_S f^2 d\mu + 2 \sum_{k=1}^{\infty} \int_S f(f \circ \tau^k) d\mu$$

converges absolutely, $\int_S S(t)^2 d\mu = t\sigma^2 + O(1)$ and, if $\sigma^2 \neq 0$, the following holds:

(i) $\sup_{z \in R} |\mu((\sigma^2 t)^{-1/2} S(t) \leqq z) - (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-x^2/2) dx| = O(t^{-v})$, *for some $v > 0$.*

(ii) *Without changing its distribution, one can redefine the process $(S(t))_{t \geqq 0}$ on a richer probability space together with the standard Brownian motion $(B(t))_{t \geqq 0}$ such that*

$$|\sigma^{-1} S(t) - B(t)| = O(t^{(1/2)-\varepsilon}), \qquad \mu\text{-a.e.},$$

for some $0 < \varepsilon < 1/2$.

(iii) *The process $(S(t))_{t \geqq 0}$ satisfies the iterated log law and other properties of Brownian motion.*

ACKNOWLEDGEMENT

We are grateful to H. Proppe who found errors in earlier drafts of this paper and made many helpful suggestions.

REFERENCES

1. D. Candeloro, *Misure invariante per transformazioni in piú dimensioni*, Atti Sem. Mat. Fis. Univ. Modena **XXXV** (1987), 33–42.

2. H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

3. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhauser, 1984.

4. F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982), 119–140.

5. C. T. Ionescu Tulcea and G. Marinescu, *Théorie ergodique pour des classes d'opérations non completement continues*, Ann. of Math. **52** (1950), 140–147.

6. V. V. Ivanov and A. G. Kachurowski, *Absolutely continuous invariant measures for locally*

expanding transformations, Preprint No. 27, Institute of Mathematics AN USSR, Siberian Section (in Russian).

7. M. Jabłoński, *On invariant measures for piecewise C^2-transformations of the n-dimensional cube*, Ann. Polon. Math. **XLIII** (1983), 185–195.

8. G. Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morcaux d'une région bornée du plan*, C.R. Acad. Sci. Paris **289**, Série A (1979), 625–627.

9. G. Keller, *Proprietés ergodiques des endomorphismes dilatants, C^2 par morceaux, des régions bornées du plan*, Thesis, Université de Rennes, 1979.

10. A. A. Kosyakin and E. A. Sandler, *Ergodic properties of a class of piecewise-smooth transformations of an interval*, Izv. VUZ Matematika (3)[118] (1972), 32–40. (English translation from the British Library, Translation Service.)

11. A. A. Kosyakin and E. A. Sandler, *Stochasticity of a certain class of discrete system*, translated from Automatika and Telemekhanika No. 9 (1972), 87–94.

12. K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Math. **33** (1969), 82–92.

13. A. Lasota and M. Mackey, *Probabilistic Properties of Deterministic Systems*, Cambridge University Press, 1985.

14. A. Lasota and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Am. Math. Soc. **186** (1973), 481–488.

15. M. Rychlik, *Bounded variation and invariant measures*, Studia Math. **LXXVI** (1983), 69–80.

16. F. Schweiger, *Invariant measures and ergodic properties of numbertheoretical endomorphisms*, Banach Center Publications, to appear.

17. E. Straube, *On the existence of invariant absolutely continuous measures*, Commun. Math. Phys. **81** (1981), 27–30.

18. M. Yuri, *On a Bernouilli property for multi-dimensional mappings with finite range structure*, in *Dynamical Systems and Nonlinear Oscillations*, Vol. 1 (Giko Ikegami, ed.), World Scientific, Singapore, 1986.